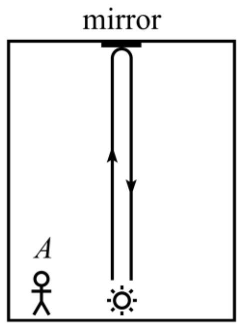
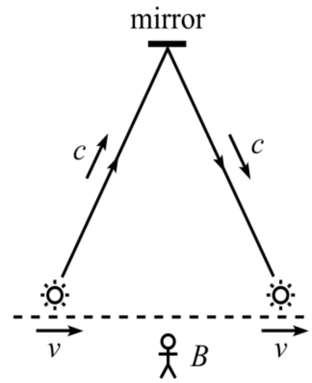
Time Dilation

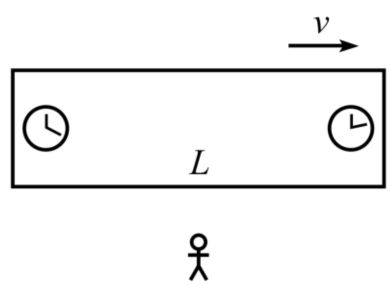


- Light moves at speed c in all frames
- Two events that occur simultaneously in one reference frame may not be simultaneous in another frame
- Two events that occur <u>at the same location</u> in frame A, separated by a time t_A, will take a greater time $t_B = \gamma\, t_A$ in frame B moving at velocity v with respect to frame A
- $\gamma = \frac{1}{\sqrt{1-\frac{v^2}{c^2}}}$ (with $\gamma \geq 1$)

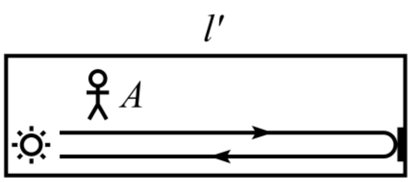


Head Start Effect

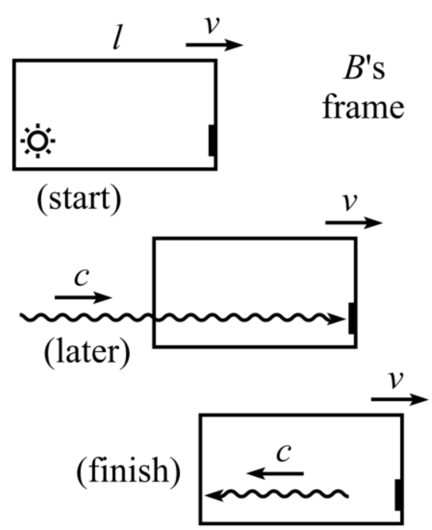
- Two clocks are positioned at the end of a train of length L (as measured in its own frame. The clocks are synchronized in the train's reference frame
- The train travels past an observer at speed v. The observer will see the *rear clock* showing a higher reading than the front clock by $\Delta t = \frac{Lv}{c^2}$
- The rear clock does not tick faster than the front clock, it simply remains a fixed time ahead of the front clock



Length Contraction



- An object in frame A moves with velocity v relative to frame B. The object has length l' along this direction of motion in A's frame. The object has a *shorter* length $l = \frac{l'}{\gamma}$ in B's frame
- Length contraction does not apply in the directions transverse to the direction of motion (set by v) between two reference frames



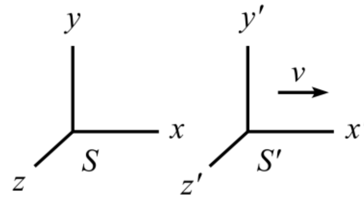
Example

A muon moves from a height h straight down towards the Earth with a large velocity v. It decays after time T in its own frame. Can the muon reach the ground?

- In the Earth's frame, time dilation applies. The muon has to travel a distance h in time $\gamma\, T$
- In the muon's frame, length contraction kicks in. The muon must travel a distance $\frac{h}{\gamma}$ in time T

Lorentz Transformation

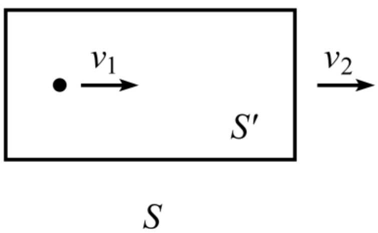
- Frame S' moves at velocity v relative to frame S
- The length and time between two events (an event is anything that has space and time coordinates) in S' are related to those in S by



$$\Delta x = \gamma(\Delta x' + v\,\Delta t') \qquad \Delta x' = \gamma(\Delta x - v\,\Delta t)$$
$$\Delta t = \gamma\left(\Delta t' + \frac{v}{c^2}\,\Delta x'\right) \qquad \Delta t' = \gamma\left(\Delta t - \frac{v}{c^2}\,\Delta x\right)$$

- Since there is no transverse length contraction, $\Delta y = \Delta y'$ and $\Delta z = \Delta z'$
- $\Delta x^2 - c^2\Delta t^2$ has the same value *in all reference frames*

Velocity Addition



- An object moves at speed v_1 with respect to frame S'. Frame S' moves at speed v_2 with respect to frame S in the same direction of motion as the object. The speed u of the object in frame S is

$$u = \frac{v_1 + v_2}{1 + \frac{v_1 v_2}{c^2}}$$

- An object moving at speed c in one reference frame moves at speed c in all reference frames
- An object moving slower than c in one reference frame moves slower than c in all reference frames

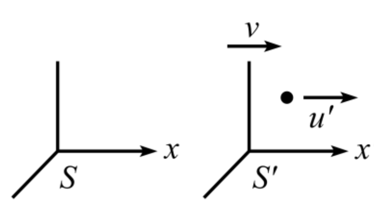
Momentum and Energy

- The relativistic energy and momentum of a particle moving with velocity $\vec{v}$ are given by

$$\vec{p} = \gamma m\vec{v} \qquad E = \gamma mc^2$$

Slow speeds ($v \ll c$): $\vec{p} \approx m\vec{v}$, $E \approx mc^2 + \frac{1}{2}m\,v^2$

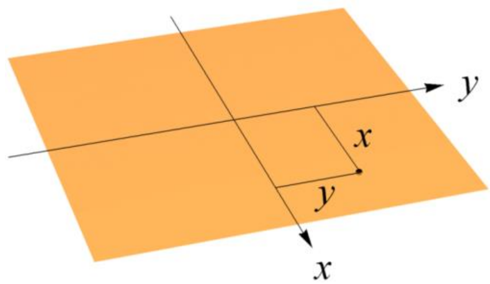
- A particle moves at speed u' in frame S', which moves at speed v relative to frame S. Momentum ($\gamma_{u'}mu'$) and energy ($\gamma_{u'}mc^2$) in S' is related to that in S by the same Lorentz transformation as space and time (with $\gamma \equiv \gamma_v$)



$$E = \gamma(E' + v\,p') \qquad E' = \gamma(E - v\,p)$$
$$p = \gamma\left(p' + \frac{v}{c^2}E'\right) \qquad p' = \gamma\left(p - \frac{v}{c^2}E\right)$$

- Energy and momentum are conserved in collisions *within the same reference frame*
- The invariant mass formula $E^2 - p^2c^2 = m^2c^4$ applies any mass *in any reference frame*

Cartesian Coordinates



An oldie but a goodie, yet not always the best choice!

Area of a circle in Cartesian coordinates

$$\int_{-R}^{R}\int_{-\sqrt{R^2-y^2}}^{\sqrt{R^2-y^2}} dx\,dy \overset{pain}{\Longrightarrow} \pi R^2$$

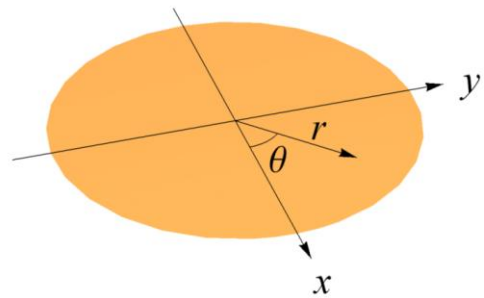
Area of a circle in Polar coordinates

$$\int_{0}^{2\pi}\int_{0}^{R} r\,dr\,d\theta \overset{easy}{\Longrightarrow} \pi R^2$$

Area Element

$$dV = dx\,dy$$

Polar Coordinates



Polar to Cartesian

$$\begin{bmatrix} x = r\,\text{Cos}[\theta] \\ y = r\,\text{Sin}[\theta] \end{bmatrix}$$

Cartesian to Polar

$$\begin{bmatrix} r = \sqrt{x^2+y^2} \\ \theta = \text{ArcTan}\left[\frac{y}{x}\right] \end{bmatrix}$$

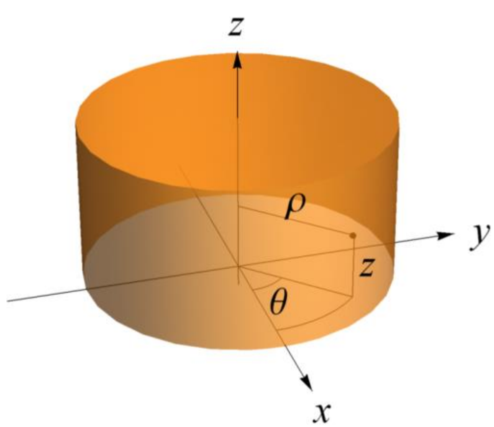
Unit Vectors

$$\begin{bmatrix} \hat{x} = \text{Cos}[\theta]\,\hat{\rho} - \text{Sin}[\theta]\,\hat{\theta} \\ \hat{y} = \text{Sin}[\theta]\,\hat{\rho} + \text{Cos}[\theta]\,\hat{\theta} \end{bmatrix} \begin{bmatrix} \hat{r} = \text{Cos}[\theta]\,\hat{x} + \text{Sin}[\theta]\,\hat{y} \\ \hat{\theta} = -\text{Sin}[\theta]\,\hat{x} + \text{Cos}[\theta]\,\hat{y} \end{bmatrix}$$

Area Element

$$dV = r\,dr\,d\theta$$

Cylindrical Coordinates

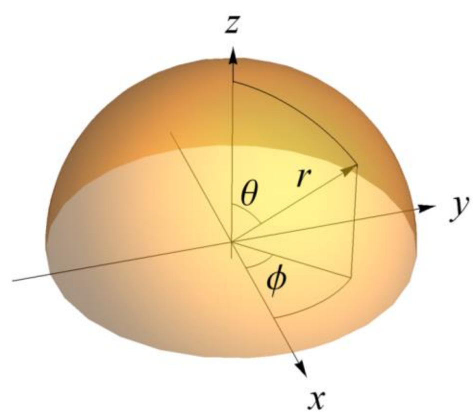


$$\begin{bmatrix} x = \rho\,\text{Cos}[\theta] \\ y = \rho\,\text{Sin}[\theta] \\ z = z \end{bmatrix} \quad \begin{bmatrix} \rho = \sqrt{x^2+y^2} \\ \theta = \text{ArcTan}\left[\frac{y}{x}\right] \\ z = z \end{bmatrix}$$

Volume Element

$$dV = r^2\,\text{Sin}[\theta]\,dr\,d\theta\,d\phi$$

Spherical Coordinates



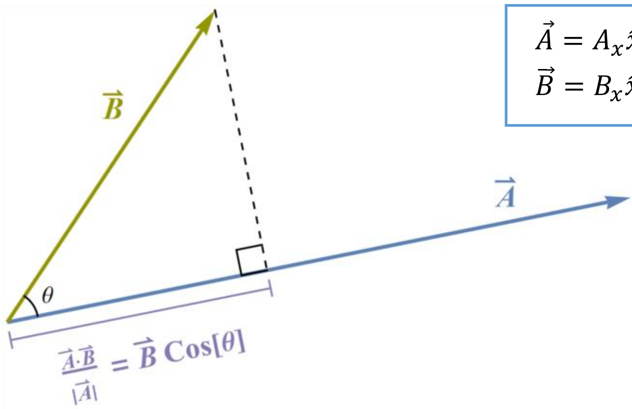
$$\begin{bmatrix} x = r\,\text{Sin}[\theta]\,\text{Cos}[\phi] \\ y = r\,\text{Sin}[\theta]\,\text{Sin}[\phi] \\ z = r\,\text{Cos}[\theta] \end{bmatrix} \begin{bmatrix} r = \sqrt{x^2+y^2+z^2} \\ \theta = \text{ArcTan}\left[\frac{(x^2+y^2)^{1/2}}{z}\right] \\ \phi = \text{ArcTan}\left[\frac{y}{x}\right] \end{bmatrix}$$

Volume Element

$$dV = r^2\,\text{Sin}[\theta]\,dr\,d\theta\,d\phi$$

Dot Product

$$\vec{A}\cdot\vec{B} = A_xB_x + A_yB_y + A_zB_z$$
$$= |\vec{A}||\vec{B}|\text{Cos}[\theta]$$

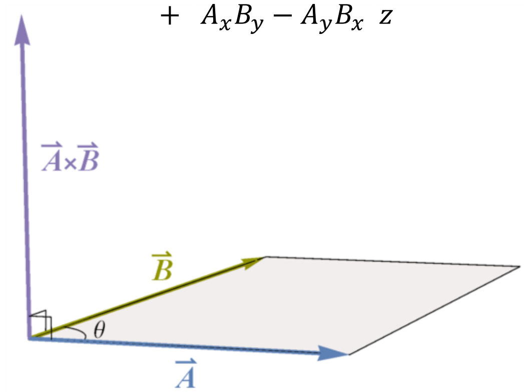


$$\vec{A} = A_x\hat{x} + A_y\hat{y} + A_z\hat{z}$$
$$\vec{B} = B_x\hat{x} + B_y\hat{y} + B_z\hat{z}$$

- $\vec{A}\cdot\vec{B}$ is a *scalar*
- $\vec{A}\cdot\vec{B} = \vec{B}\cdot\vec{A} \longrightarrow \vec{A}\cdot\vec{A} = |\vec{A}|^2$
- For any *unit* vector $\hat{n}$, $\vec{A}\cdot\hat{n}$ represents the length of $\vec{A}$ along the direction $\hat{n}$

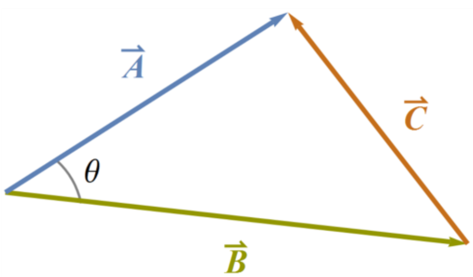
Cross Product

$$\vec{A}\times\vec{B} = (A_yB_z - A_zB_y)\hat{x}$$
$$+ \ A_zB_x - A_xB_z \ y$$
$$+ \ A_xB_y - A_yB_x \ z$$



- $\vec{A}\times\vec{B}$ is a *vector*
- $\vec{A}\times\vec{B} = -\vec{B}\times\vec{A} \longrightarrow \vec{A}\times\vec{A} = \vec{0}$
- Direction given by right-hand rule and magnitude $|\vec{A}\times\vec{B}| = |\vec{A}||\vec{B}|\text{Sin}[\theta]$ equal to parallelogram area

Law of Cosines



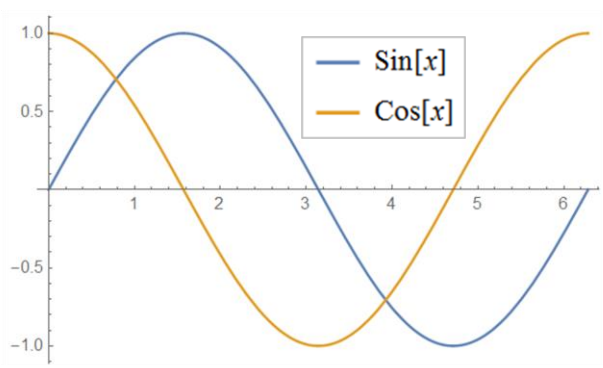
- Triangle defined by vectors $\vec{A}$ and $\vec{B}$
- Thirds leg given by $\vec{C} = \vec{A} - \vec{B}$

$$|\vec{\boldsymbol{C}}|^2 = |\vec{\boldsymbol{A}}|^2 + |\vec{\boldsymbol{B}}|^2 - 2|\vec{\boldsymbol{A}}||\vec{\boldsymbol{B}}|\textbf{Cos}[\boldsymbol{\theta}]$$

Proof

$$\vec{C}\cdot\vec{C} = (\vec{A}-\vec{B})\cdot(\vec{A}-\vec{B})$$
$$= |\vec{A}|^2 + |\vec{B}|^2 - 2\,\vec{A}\cdot\vec{B}$$

Trig Functions



$$\text{Cos}[0] = 1 \qquad \text{Sin}[0] = 0$$
$$\text{Cos}\left[\frac{\pi}{6}\right] = \frac{\sqrt{3}}{2} \qquad \text{Sin}\left[\frac{\pi}{6}\right] = \frac{1}{2}$$
$$\text{Cos}\left[\frac{\pi}{4}\right] = \frac{\sqrt{2}}{2} \qquad \text{Sin}\left[\frac{\pi}{4}\right] = \frac{\sqrt{2}}{2}$$
$$\text{Cos}\left[\frac{\pi}{3}\right] = \frac{1}{2} \qquad \text{Sin}\left[\frac{\pi}{3}\right] = \frac{\sqrt{3}}{2}$$
$$\text{Cos}\left[\frac{\pi}{2}\right] = 0 \qquad \text{Sin}\left[\frac{\pi}{2}\right] = 1$$
$$\text{Cos}[\pi - x] = -\text{Cos}[x] \qquad \text{Sin}[\pi - x] = \text{Sin}[x]$$

$$\text{Sin}[x] = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \cdots = \sum_{j=0}^{\infty}(-1)^j\frac{x^{2j+1}}{(2j+1)!} = \frac{e^{ix} - e^{-ix}}{2}$$

$$\text{Cos}[x] = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \cdots = \sum_{j=0}^{\infty}(-1)^j\frac{x^{2j}}{(2j)!} = \frac{e^{ix} + e^{-ix}}{2}$$

$$\text{Sin}[x+y] = \text{Sin}[x]\text{Cos}[y] + \text{Cos}[x]\text{Sin}[y]$$
$$\text{Cos}[x+y] = \text{Cos}[x]\text{Cos}[y] - \text{Sin}[x]\text{Sin}[y]$$
$$\text{Sin}[2x] = 2\text{Sin}[x]\text{Cos}[x]$$
$$\text{Cos}[2x] = \text{Cos}[x]^2 - \text{Sin}[x]^2$$
$$= 2\text{Cos}[x]^2 - 1 = 1 - 2\text{Sin}[x]^2$$

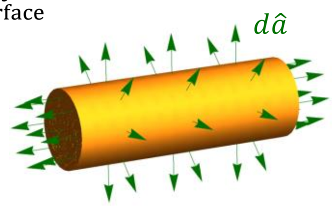
Surface Integrals

$$\oint_{\text{surface}} da \qquad \oint_{\text{surface}} f[\vec{r}]da$$

Integrate over infinitesimal area elements

$$\oint_{\text{surface}} \vec{F}[\vec{r}]\cdot d\vec{a} = \oint_{\text{surface}} (\vec{F}[\vec{r}]\cdot d\hat{a})da$$

$d\hat{a}$ is the normal direction (perpendicular to a surface)



Sphere of radius R centered at the origin

$$\oint_{\text{surface}} da = \int_0^{2\pi}\int_0^{\pi} R^2\text{Sin}[\theta]\, d\theta\, d\phi$$

$$\oint_{\text{surface}} f[\vec{r}]da = \int_0^{2\pi}\int_0^{\pi} f[\vec{r}]R^2\text{Sin}[\theta]\, d\theta\, d\phi$$

$$\oint_{\text{surface}} \vec{F}[\vec{r}]\cdot d\vec{a} = \int_0^{2\pi}\int_0^{\pi} (\vec{F}[\vec{r}]\cdot\hat{r})R^2\text{Sin}[\theta]\, d\theta\, d\phi$$

Gauss's Law

$\vec{E}$ = Electric field

$\epsilon_0 = \frac{1}{4\pi k}$

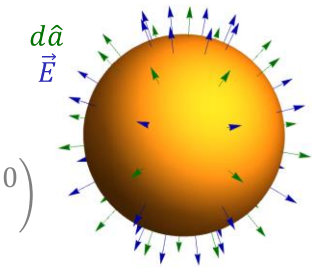
$d\vec{a}$ = Normal vector of infinitesimal area

Q_{in} = Charge enclosed in surface

$$\oint_{\text{surface}} \vec{E}[\vec{r}]\cdot d\vec{a} = \frac{Q_{\text{in}}}{\epsilon_0}$$

Choose Gaussian surface that utilizes symmetry

(on every surface $\vec{E}\cdot d\vec{a} = 0$ or $\vec{E}\cdot d\vec{a} = E\, da$)



Electric field from point charge q at the origin

Radial symmetry $\Longrightarrow$ $\vec{E}[\vec{r}] = E[r]\hat{r}$

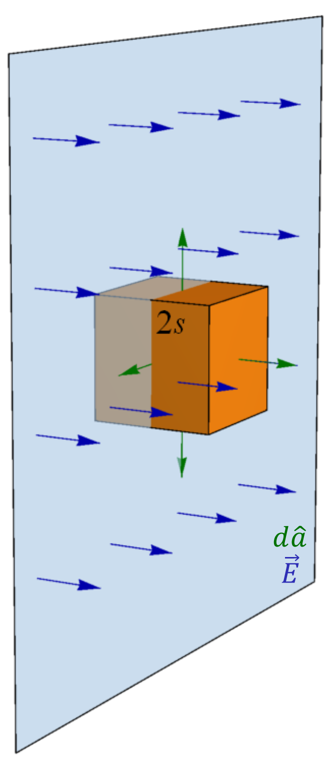
($\vec{E}$ points radially and only depends on distance r from origin)

Gaussian sphere $\Longrightarrow$ $d\vec{a} = \hat{r}\, da$

$$\oint \vec{E}[\vec{r}]\cdot d\vec{a} = \oint E[r]da = E[r]\oint da = E[r](4\pi r^2) = \frac{q}{\epsilon_0}$$

$$\vec{E}[\vec{r}] = \frac{q}{4\pi\epsilon_0 r^2}\hat{r}$$

Example: Infinite Sheet of Charge



Infinite plane at $x = 0$ with uniform charge density σ

Translational symmetry $\Longrightarrow$ $\vec{E}[\vec{r}] = E[x]\hat{x}$

($\vec{E}$ points away from plane and only depends on distance x from plane)

Gaussian cube with side length $2s$ centered on the plane

- $E[x]\hat{x}\cdot d\vec{a} = 0$ for all sides that intersect the plane
- $E[x]\hat{x}\cdot d\vec{a} = E[x]da$ for both surfaces parallel to plane

$$\oint \vec{E}[\vec{r}]\cdot d\vec{a} = 2E[x](4s^2) = \frac{\sigma(4s^2)}{\epsilon_0}$$

$$\vec{E}[\vec{r}] = \begin{cases} \frac{\sigma}{2\epsilon_0}\hat{x} & x > 0 \\ -\frac{\sigma}{2\epsilon_0}\hat{x} & x < 0 \end{cases}$$

Exactly on the plane, electric field must be zero by symmetry!

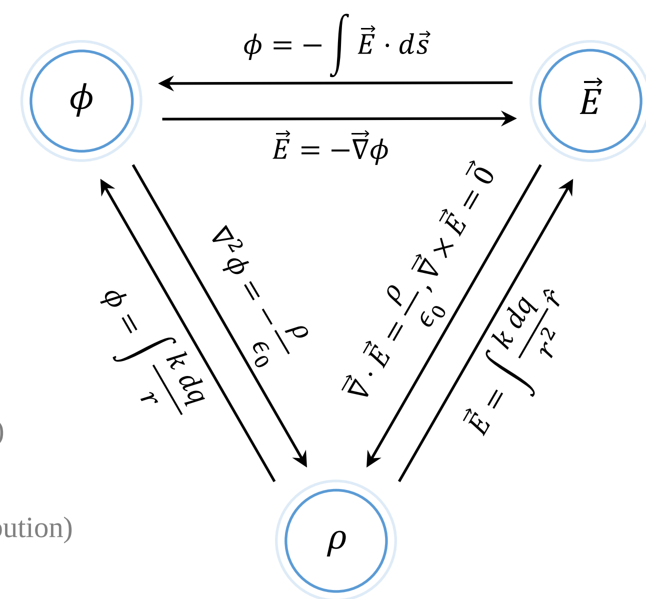
Discontinuity in $\vec{E}$ across a sheet of charge is always $\frac{\sigma}{\epsilon_0}$

Value of $\vec{E}$ on the sheet is the average of its values on both sides

Electrostatics

Fundamental quantities:

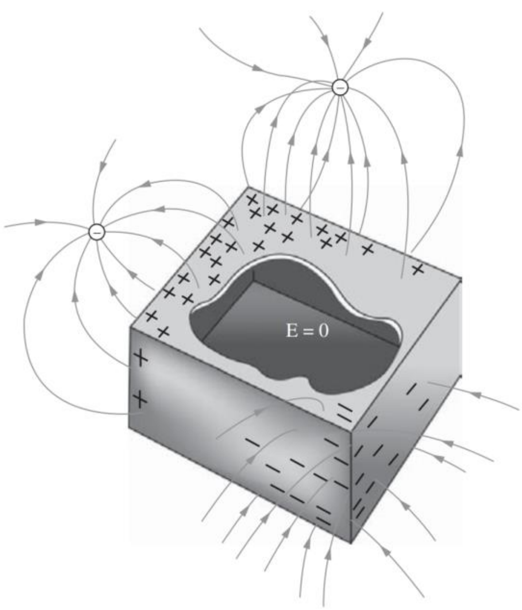
- ρ – Charge distribution
- $\vec{E}$ – Force per unit charge on a test particle
- ϕ – Potential energy per unit charge to place a test particle



Work required to assemble charge distribution:

$$U = \frac{1}{2}\sum_{i=1}^{N}\sum_{j\neq i}\frac{k\,q_i q_j}{r_{jk}} \qquad \text{(Point charges)}$$

$$U = \frac{\epsilon_0}{2}\int E^2 dv = \frac{1}{2}\int \rho\,\phi\,dv \qquad \text{(Charge distribution)}$$



Conductors

Key properties

- $\vec{E} = \vec{0}$ inside a conductor
- $\rho = 0$ inside a conductor
- Any net charge resides on the surface
- A conductor surface is equipotential
- $\vec{E}$ perpendicular to surface just outside a conductor

Example: Parallel Plates
(Ignoring Edge Effects)

- Uniform charge density $\sigma = \frac{Q}{A}$ spread over both inner surfaces
- $\vec{E} = \frac{\sigma}{\epsilon_0}(-\hat{z})$ between plates, $\vec{E} = \vec{0}$ outside plates

$$C \equiv \frac{Q}{\phi_1 - \phi_2} = \frac{\epsilon_0 A}{s} \qquad \text{(Capacitance)}$$

$$U \equiv \frac{Q^2}{2C} = \frac{\epsilon_0 E^2}{2} A s \qquad \left(\begin{array}{c}\text{Energy Stored}\\ \text{in Capacitor}\end{array}\right)$$